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LETTER TO THE EDITOR

Comment on the q-deformed fermionic oscillator

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Abstract. The q-deformed fermionic oscillator is identified with the usual fermionic oscillator by virtue of a simple transformation. So the realization of quantum algebra $su_q(2)$ can also be constructed from the usual fermionic oscillator.

In recent years the development of the quantum inverse method and the study of solutions of the Yang–Baxter equation have led to the concept of quantum groups and algebras [1]. It was also found that the quantum groups have important applications in exactly solvable statistical models [2] and in 2D conformal field theories [3]. More recently, a new realization of the quantum algebra $su_q(2)$ has been obtained by introducing a q-analogue of the usual bosonic harmonic oscillator and making the Jordan–Schwinger mapping [4]. Naturally, besides the q-deformed bosonic oscillator, *a q-deformed fermionic equivalent has also been introduced to construct the oscillator* representation of the q-deformed superalgebras [5], some q-deformed classical Lie algebras A_{n-1}, B_n, C_n and D_n [6], and quantum exceptional algebras [7].

In this letter, we present an explicit proof to show that the well known q-deformed fermionic oscillator is nothing but the usual fermionic oscillator. Of course, using the standard techniques of contraction of Lie algebras and group theory, one can derive the q-oscillator algebras, both bonsonic and fermionic, from the corresponding quantum algebras [8]. Different contractions of simplest rank-one quantum Lie (super)algebras $sl_q(2)$ and $osp_q(1/2)$ give different generalizations of the harmonic oscillator, i.e. q-deformed oscillators. The q-deformed fermionic oscillator, discussed in this letter, is the most commonly accepted one by various authors [5–7, 9].

We start from the most commonly accepted definition of the q-deformed fermionic oscillator

$$\tilde{a}\tilde{a}^{+} + q\tilde{a}^{+}\tilde{a} = q^{\tilde{N}} \tag{1}$$

$$\tilde{a}\tilde{a}^{+} + q^{-1}\tilde{a}^{+}\tilde{a} = q^{-\tilde{N}} \tag{2}$$

$$\tilde{a}^{2} = \tilde{a}^{+2} = 0. \tag{3}$$

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where $\tilde{a}$, $\tilde{a}^+$ and $\tilde{N}$ are the q-deformed fermionic annihilation, creation and number operators, respectively, which also satisfy

$$[\tilde{N}, \tilde{a}] = -\tilde{a} \qquad [\tilde{N}, \tilde{a}^+] = -\tilde{a}^+. \tag{4}$$

From (1) and (2), it follows that

$$[\tilde{N}] \equiv \frac{q^{\tilde{N}} - q^{-\tilde{N}}}{q - q^{-1}} = \tilde{a}^+ \tilde{a}. \tag{5}$$

Considering the following transformation

$$a = q^{-\tilde{N}/2} \tilde{a} \qquad a^+ = \tilde{a}^+ q^{-\tilde{N}/2} \tag{6}$$

it is obvious that

$$[\tilde{N}, a] = -a \qquad [\tilde{N}, a^+] = a^+ \tag{7}$$

which implies

$$q^{\tilde{N}/2} a q^{-\tilde{N}/2} = q^{-1/2} a \qquad q^{\tilde{N}/2} a^+ q^{-\tilde{N}/2} = q^{1/2} a^+.$$

Thus from (1) and (3) we obtain

$$aa^+ + a^+ a = 1 \qquad a^2 = a^{+2} = 0. \tag{8}$$

This means that a and a^+ are the usual fermionic oscillator. Now, we define a new operator

$$N \equiv a^+ a. \tag{9}$$

From (7), we have

$$N^2 = N \tag{10}$$

and

$$[N, a] = -a \qquad [N, a^+] = a^+. \tag{11}$$

Using (6) we can rewrite $[\tilde{N}]$ as

$$[\tilde{N}] = a^+ q^{\tilde{N}} a = q^{\tilde{N}-1} a^+ a = q^{\tilde{N}-1} N$$

or

$$\frac{1 - q^{-2\tilde{N}}}{1 - q^{-2}} = N. \tag{12}$$

Equation (12) leads to

$$-2\tilde{N} = \frac{1}{\ln q} \ln[1 - (1 - q^{-2}) N] \tag{13}$$

where the right-hand side of (13) is understood as the series expansion of $\ln(1-x)$. Because of (10), we have

$$\begin{aligned} -2\tilde{N} &= \frac{1}{\ln q}\left[(q^{-2} - 1)N - \tfrac{1}{2}(q^{-2}-1)^2 N + \tfrac{1}{3}(q^{-2}-1)^3 N - \dots\right] \\ &= \frac{N}{\ln q} \ln(1 + q^{-2} - 1) = -2N. \end{aligned} \tag{14}$$

It follows from (14) that

$$\tilde{N} = N. \tag{15}$$

Also from (10), it is easy to see

$$q^N = 1 + (q-1)N. \tag{16}$$

Thus the transformation (6) becomes

$$\tilde{a} = q^{\tilde{N}/2}a = q^{N/2}a = [1 + (q^{1/2} - 1)N]a = a + (q^{1/2} - 1)Na = a \tag{17}$$

and similarly,

$$\tilde{a}^+ = a^+. \tag{17'}$$

Equations (17) and (17′) clearly show that the so-called 'q-deformed fermionic oscillator', defined by (1) and (3), is nothing but the usual fermionic oscillator.

Substituting the transformation (6) into (2) gives

$$aa^+ + q^{-2}a^+a = q^{-2\tilde{N}} = q^{-2N}. \tag{18}$$

Also from (16), we can rewrite (18) as

$$aa^+ + q^{-2}a^+a = 1 - N + q^{-2}N = 1 - a^+a + q^{-2}a^+a$$

which leads again to

$$aa^+ + a^+a = 1.$$

Conversely, we can also rewrite the usual fermionic oscillator in terms of the q-deformed formula:

$$aa^+ + qa^+a = 1 - N + qN = q^N \tag{19}$$

and

$$aa^+ + q^{-1}a^+a = 1 - N + q^{-1}N = q^{-N}. \tag{20}$$

It tells us again that there is no difference between the q-deformed and the usual fermionic oscillator.

Another result from (16) is

$$[N] = \frac{q^N - q^{-N}}{q - q^{-1}} = \frac{1 + (q-1)N - 1 - (q^{-1} - 1)N}{q - q^{-1}} = N \tag{21}$$

which is just our expectation.

As an application of the above discussion, we give the Jordan–Schwinger realization of $SU_q(2)$ using two independent usual fermionic oscillators:

$$J_+ = a_1^+a_2 \qquad J_- = a_2^+a_1 \qquad J_0 = \tfrac{1}{2}(a_1^+a_1 - a_2^+a_2) \tag{22}$$

which satisfy the quantum commutation relations:

$$\begin{aligned} [J_+, J_-] &= [2J_0] = \frac{q^{2J_0} - q^{-2J_0}}{q - q^{-1}} \\ [J_0, J_\pm] &= \pm J_\pm. \end{aligned} \tag{23}$$

We can also construct the Jordan–Schwinger realization of $SU_q(2)$ from the one-mode usual fermionic oscillator only:

$$J_+ = a^+ \qquad J_- = a \qquad J_0 = a^+a - \tfrac{1}{2} \tag{24}$$

which again obey the quantum algebra (23). This fact means that the usual fermionic oscillator can also be used to realize the quantum algebra $SU_q(2)$.

A few words need to be said as a conclusion. The fact discussed in this letter is similar to the 2×2 representation of SU(2), where the usual generators satisfy both the $SU_q(2)$ commutation relations and the classical SU(2) relations. As far as we know, Ng first noticed the relation between the q-deformed and the usual fermionic oscillator [10], but his argument used a precondition that the number operator of the q-deformed fermionic oscillator can take on values 0 and 1 only. In this letter, our proof only starts from the most commonly accepted definition of the q-deformed fermionic oscillator and does not use any other preconditions. It is possible to construct representations of quantum exceptional algebras from the usual fermionic oscillator only. Work in this direction is in progress.

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